

.11 DENSITY AND UNIVERSALITY

14.6.2004
Martin Jaggi

Book: Stasys Jukna - Extremal Combinatorics

- Definitions
- Dense sets
 - Size of dense sets
- Universal sets
 - Isolated neighbor condition for graphs
 - Equivalence with universal sets
 - Construction of very small universal sets using Paley graphs
- Full graphs

Definitions

Projections

The **projection** of a vector $v = (v_1, \dots, v_n)$ onto a set of coordinates $S = \{i_1, \dots, i_k\}$ is the vector $v|_S := (v_{i_1}, \dots, v_{i_k})$.

The projection of a set of vectors $A \subset \{0,1\}^n$ is the set $A|_S := \{v|_S \mid v \in A\}$.

Dense and Universal Sets

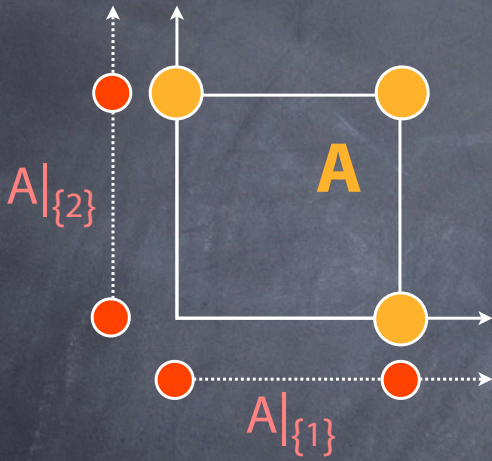
A vector set $A \subseteq \{0,1\}^n$ is called **(n,k)-universal** if the projection of A onto **any** subset S of k coordinates contains all possible 2^k configurations.

$$\forall S \quad |S| = k \quad A|_S = \{0,1\}^k$$

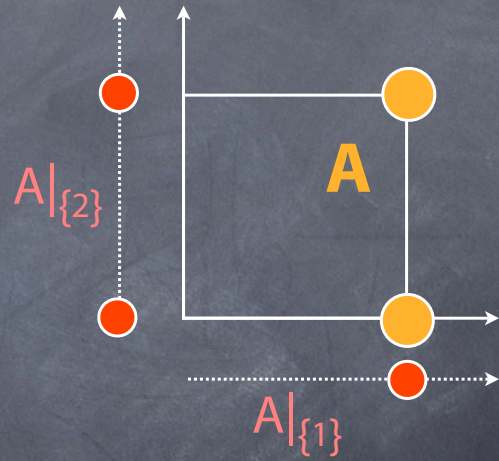
A is called **(n,k)-dense** if the same holds not necessarily for all but for **at least one** subset S of k indices.

$$\exists S \quad |S| = k \quad A|_S = \{0,1\}^k$$

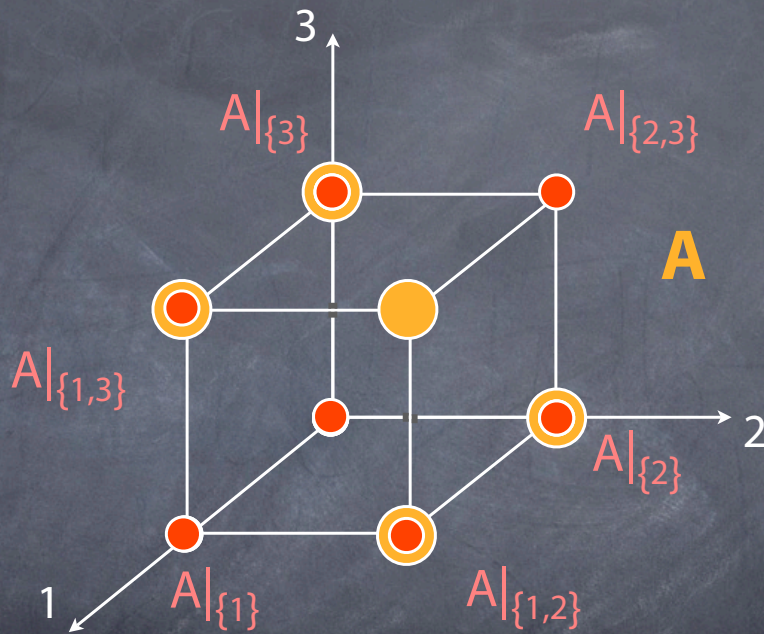
Examples



A is (2,1)-dense
and (2,1)-universal



A is (2,1)-dense
but **not** (2,1)-universal



A is (3,1)-dense
and (3,1)-universal

A is (3,2)-dense
but not (3,2)-universal

Dense sets

What can one say about the **size** of dense sets ?

• $|A| \geq 2 \Rightarrow A$ $(n,1)$ -dense

• A (n,k) -dense $\Rightarrow |A| \geq 2^k$

• There are $\sum_{i=0}^{k-1} \binom{n}{i}$ vectors in $\{0,1\}^n$ which have less than k ones. These are obviously **not** (n,k) -dense.

$$H(n,k) := \sum_{i=0}^{k-1} \binom{n}{i}$$

Theorem 1:

(Perles and Shelah 1972 / Sauer 1972 / Vapnik and Chervonenkis 1971)

$$A \subseteq \{0,1\}^n \text{ and } |A| > H(n,k) \Rightarrow A \text{ } (n,k)\text{-dense}$$

Proof

Induction on n :

Base cases:

$|A| > H(n,1) = 1$ means A contains at least two different vectors and hence is $(n,1)$ -dense.

$|A| > H(n,n) = 2^n - 1$ means A is the entire n -cube and hence is (n,n) -dense.

$n-1 \rightarrow n$:

Let **B** be the projection of A onto the first $n-1$ coordinates, and **C** the set of all vectors u in $\{0,1\}^{n-1}$ for which both vectors $(u,0)$ and $(u,1)$ belong to A.

$$|A| = |B| + |C|$$

- If $|B| > H(n-1,k)$ then B is $(n-1,k)$ -dense by induction hypothesis, and hence the whole set A is also (n,k) -dense.

- If $|B| \leq H(n-1,k)$ then

$$|C| = |A| - |B| > H(n,k) - H(n-1,k)$$

$$\begin{aligned} \binom{n}{i} &= \binom{n-1}{i-1} + \binom{n-1}{i} \\ &= \sum_{i=0}^{k-1} \binom{n}{i} - \sum_{i=0}^{k-1} \binom{n-1}{i} \\ &= \sum_{i=1}^{k-1} \binom{n-1}{i-1} \\ &= \sum_{j=0}^{k-2} \binom{n-1}{j} = H(n-1,k-1) \end{aligned}$$

So C is $(n-1,k-1)$ -dense by induction hypothesis. Because $C \times \{0,1\}$ lies in A, the whole set A is also (n,k) -dense. ▮

Universal sets

Isolated neighbor condition

Property of (bipartite) graphs which is equivalent to the universality property of 0-1 vectors.

A vector set $A \subset \{0,1\}^n$ is called **(n,k)-universal** if the projection of A onto **any** subset of k coordinates S contains all possible 2^k configurations.

$$A|_S = \{0,1\}^k \quad \forall S \quad |S| = k$$

Bipartite graphs

Given a bipartite graph with parts of size n:
 $G = (V_1, V_2, E)$ with $|V_1| = |V_2| = n$

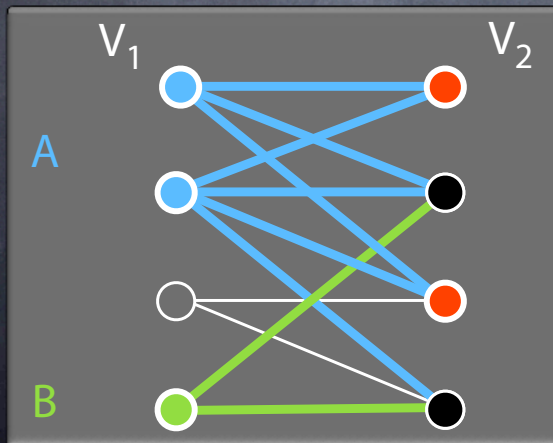
we say:

A node $x \in V_2$ is a **common neighbor** for a set of nodes $A \subseteq V_1$ if x is jointed to **each** node of A.

A node $x \in V_2$ is a **common non-neighbor** for a set of nodes $B \subseteq V_1$ if x is jointed to **no** node of B.

$$A, B \subseteq V_1$$

$v(A, B) := \#$ nodes in V_2 that are
common neighbors of A
and common non-neighbors of B
at the same time.



$$v(A, B) = 2$$

(Gál 1998)

A bipartite graph $G = (V_1, V_2, E)$ satisfies the **isolated neighbor condition for k** if

$$v(A, B) > 0$$

for any two disjoint subsets $A, B \subseteq V_1$ such
that $|A| + |B| = k$.

Proposition 2:

Let G be a bipartite Graph with parts of size n
and C be the set of columns of its adjacency
matrix. then:

G satisfies the isolated neighbor condition for k
 $\Leftrightarrow C$ is (n, k) -universal.

Proof of Proposition 2:

Let $M = (m_{x,y})$ be the adjacency matrix of G .

$$m_{x,y} = \begin{cases} 1 & (x,y) \in E \\ 0 & \text{otherwise} \end{cases}$$

\Rightarrow

nodes of V_1

Let $S = \{i_1, \dots, i_k\}$ be a subset of k rows of M .

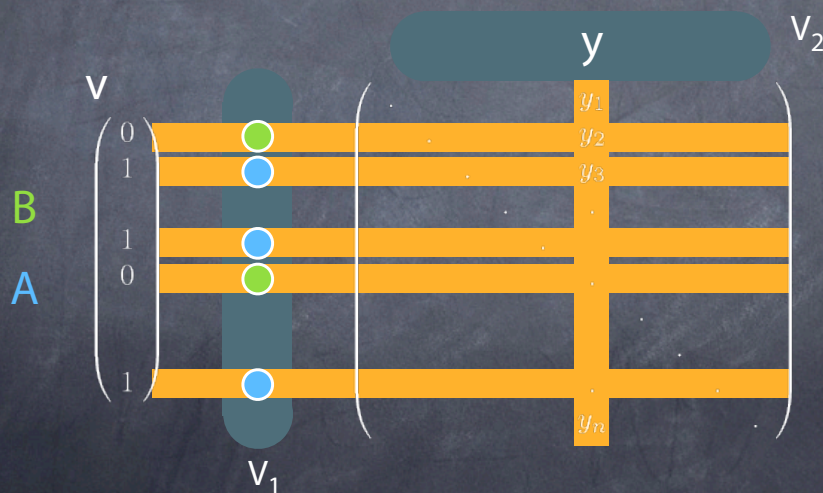
Let $v = (v_{i_1}, \dots, v_{i_k})$ be an arbitrary vector in $\{0,1\}^k$.

Is there a column y of M with $y|_S = v$?

Split our S into two subsets A and B

$j \in A$ iff $v_j = 1$, $j \in B$ iff $v_j = 0$

$$|A| + |B| = |S| = k$$



$\exists y|_S = v$

Construction of small universal sets

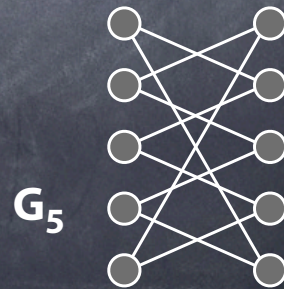
Paley graphs

A bipartite **Paley graph** is a bipartite graph

$G_p = (V_1, V_2, E)$ with parts $V_1 = V_2 = \mathbb{F}_p$.

$(x, y) \in E$ iff $x - y$ is a non-zero square in \mathbb{F}_p .

Where p is a prime number.



Theorem 3:

(Bollobás and Thomason 1981)

Let $G_p = (V_1, V_2, E)$ be a bipartite Paley graph with $p \geq 9$, and A, B disjoint sets of nodes in V_1 such that $|A| + |B| = k$. then

$$|v(A, B) - 2^{-k}p| \leq k\sqrt{p}$$

in particular, $v(A, B) > 0$ as long as $k2^k < \sqrt{p}$
 $\Leftrightarrow k\sqrt{p} < 2^{-k}p$

Quadratic residue character:

$$\chi(x) := \begin{cases} 1 & \exists a \in \mathbb{F}_p^* \quad x = a^2 \\ 0 & x = 0 \\ -1 & \nexists a \in \mathbb{F}_p \quad x = a^2 \end{cases} \quad x \in \mathbb{F}_p$$

$$\chi(x) = x^{(p-1)/2} \quad \forall x \in \mathbb{F}_p$$

$$\chi(x \cdot y) = \chi(x) \cdot \chi(y) \quad \forall x, y \in \mathbb{F}_p$$

Theorem (Weil 1948):

Let $f(x)$ be a polynomial over \mathbb{F}_p which is not the square of another polynomial and has precisely s distinct zeros. then:

$$\left| \sum_{x \in \mathbb{F}_p} \chi(f(x)) \right| \leq (s-1)\sqrt{p}$$

Proof of Theorem 3:

Remember that (x, y) is an edge in $G_p \iff \chi(x-y) = 1$.

We say $x' \in V_2$ is the **copy** of $x \in V_1$ if both these nodes correspond to the same element in \mathbb{F}_p .

No x is joined to its copy x' .

We define

$$\mathbf{U} := V_2 - (A' \cup B') \quad \text{where } A', B' \subseteq V_2 \text{ are the copies of } A, B \subseteq V_1$$

$$g(x) := \prod_{a \in A} (1 + \chi(x - a)) \prod_{b \in B} (1 - \chi(x - b)) \quad \text{for nodes } x \in V_2$$

For each node $x \in U$, $g(x)$ is non-zero iff x is joined to every node in A and to no node in B , in which case it is precisely 2^k . So

$$\sum_{x \in U} g(x) = 2^k v^*(A, B)$$

where $v^*(A, B)$ is the number of nodes in U which are joined to every node of A and to no node of B .

$$g(x) = \prod_{c \in A \cup B} (1 \pm \chi(x - c))$$

$$= 1 + \sum_{C \subset A \cup B} (-1)^{|C \cap B|} \chi(f_C(x))$$

$$\text{where } f_C(x) = \prod_{c \in C} (x - c)$$

$$\begin{aligned} \left| \sum_{x \in \mathbb{F}_p} g(x) - p \right| &= \left| \sum_{x \in \mathbb{F}_p} (g(x) - 1) \right| = \left| \sum_{x \in \mathbb{F}_p} \sum_C (-1)^{|C \cap B|} \chi(f_C(x)) \right| \\ &\leq \sum_C \left| \sum_{x \in \mathbb{F}_p} (-1)^{|C \cap B|} \chi(f_C(x)) \right| \leq \sum_C \left| \sum_{x \in \mathbb{F}_p} \chi(f_C(x)) \right| \\ &\leq \sum_C (|C| - 1) \sqrt{p} \quad \text{(Weil's theorem)} \\ &= \sqrt{p} \sum_{s=1}^k \binom{k}{s} (s - 1) = \sqrt{p} \left(\sum_{s=1}^k \binom{k}{s} s - \binom{k}{s} \right) \\ &= \sqrt{p} (k 2^{k-1} - (2^k - 1)) \end{aligned}$$

$$\sum_{s=1}^k s \binom{k}{s} = k 2^{k-1}$$

Because for every $x \in A' \cup B'$, $g(x) \leq 2^{k-1}$, or we have

$$\left| \sum_{x \in A' \cup B'} g(x) \right| \leq k 2^{k-1}$$

$$\left| \sum_{x \in U} g(x) - p \right| \leq \sqrt{p} (k 2^{k-1} - 2^k + 1) + k 2^{k-1}$$

By dividing both sides by 2^k and using $\left(\sum_{x \in U} g(x) = 2^k v^*(A, B) \right)$ we get:

$$|v^*(A, B) - 2^{-k} p| \leq \sqrt{p} \left(\frac{k}{2} - 1 + \frac{1}{2^k} \right) + \frac{k}{2}$$

and because $|v(A, B) - v^*(A, B)| \leq |A' \cup B'| = k$:

$$|v(A, B) - 2^{-k} p| \leq \frac{k\sqrt{p}}{2} - \sqrt{p} + \frac{\sqrt{p}}{2^k} + \frac{k}{2} + k$$

for $p > 9$

$$\leq k\sqrt{p}$$

Theorem 3 together with Proposition 2 gives us,
 for every prime $n > 9$
 and for every k such that $k 2^k < \sqrt{n}$,
 an explicit construction of (n, k) -universal sets of
 size n .

Using linear codes it is possible to construct
 such sets of size $n^{O(k)}$ for arbitrary k .

Full graphs

A graph containing of order n is called **k-full** iff it contains every k -vertex graph as a subgraph.

How many vertices a graph must have to be k -full ?

If G is a k -full graph of order n then $\binom{n}{k}$ is at least the number of non-isomorphic subgraphs of order k .

$$\binom{n}{k} \geq \frac{2^{\binom{k}{2}}}{k!} \Leftrightarrow n(n-1)\dots(n-k+1) \geq 2^{\frac{k(k-1)}{2}}$$
$$\Rightarrow n \geq 2^{\frac{(k-1)}{2}}$$

Construction of small k -full graphs

Let P_k be the graph of order $n = 2^k$ whose vertices are the subsets of $\{1, \dots, k\}$, and where two vertices are joined iff $|A \cap B|$ is even. Exception: A is joined to $\{\}$ iff $|A|$ is even.

Theorem 4: (Bollobás and Thomason 1981)

The graph P_k is k -full.

Proof of Theorem 4:

Let G be an arbitrary graph with vertex set $\{v_1, v_2, \dots, v_k\}$. We claim there are sets A_1, A_2, \dots, A_k uniquely determined by G such that

$$|A_i \cap A_j| \text{ is even iff } v_i \text{ and } v_j \text{ are joined in } G$$
$$A_i \subseteq \{1, \dots, i\}$$

Set $A_1 := \{1\}$

Suppose we already have chosen A_1, A_2, \dots, A_{i-1} .

We search for an A_i which is properly joined to all the sets A_1, A_2, \dots, A_{i-1} , that is, $|A_i \cap A_j|$ must be even iff v_i is joined to v_j in G .

We will obtain A_i as the last set in the sequence $B_1^i \subseteq B_2^i \subseteq \dots \subseteq B_{i-1}^i = A_i$ with, for each $1 \leq j < i$, B_j^i is a set properly joined to all sets A_1, A_2, \dots, A_j .

$$B_0^i := \{i\}$$

For $j \geq 1$:

If v_j is joined to v_i set $B_j^i := B_{j-1}^i$ or $B_j^i := B_{j-1}^i \cup \{j\}$ depending on $|B_{j-1}^i \cap A_j|$.

If v_j is not joined to v_i we act dually.

Our choice of whether j is in B_j^i does effect

$|B_j^i \cap A_j|$ (since $j \in A_j$),

but none of $|B_j^i \cap A_k|$ $k < j$ (since $A_k \subseteq \{1, \dots, k\}$)

The End