New Results in Tropical Discrete Geometry

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Abstract

Following the recent work of Develin and Sturmfels and others (see, e.g., [10, 16, 2, 11]), we investigate discrete geometric questions over the tropical semiring $(\mathbb{R}, \min, +)$. Specifically, we obtain the following tropical analogues of classical theorems in convex geometry: a separation theorem for a pair of disjoint tropical polytopes by tropical halfspaces and tropical versions of Radon's lemma, Helly's theorem, the Centerpoint theorem, and Tverberg's theorem, including algorithms to find tropical centerpoints and Tverberg points. We also prove tropical analogues of the colored Carathéodory and colored Tverberg theorems. Furthermore, we study the tropical analogues of k-sets and levels in halfspace arrangements and obtain tight bounds of $\Theta(n^{d-1})$ for the number of tropical halving sets in any fixed dimension d.

1 Introduction

In tropical mathematics, the basic object of study is the **tropical semiring**[1] $(\mathbb{R}, \oplus, \odot)$, also referred to as the **min-plus algebra**. This is the set of real numbers with the arithmetic operations of **tropical addition** and **tropical multiplication**, defined by

$$a \oplus b := \min\{a, b\} \qquad \text{and} \qquad a \odot b := a + b,$$

respectively.

The d-dimensional space $\mathbb{R}^d$ with component-wise **tropical vector addition** and **tropical tropical scalar multiplication**, defined by $(a_1, \ldots, a_d) \oplus (b_1, \ldots, b_d) := (a_1 \oplus b_1, \ldots, a_d \oplus b_d)$ and $\lambda \odot (a_1, \ldots, a_d) := (\lambda \odot a_1, \ldots, \lambda \odot a_n)$, respectively, is a **semimodule** over the semiring $(\mathbb{R}, \oplus, \odot)$.

These structures occur naturally in many contexts. A well-known example is the all-pairs shortest path problem in graphs, which can be viewed in terms of matrix multiplication over the tropical semiring.[2] For more background on linear algebra and matrix theory over the tropical and other idempotent semirings, and further applications in graph theory, automata theory, scheduling, control theory, and other areas, see, e.g., the surveys [23, 13, 6] and the monographs [3, 14]. For a quick elementary introduction to tropical mathematics, see [25].

Tropical Discrete Geometry. The notion of convexity in general idempotent semimodules was first investigated by Zimmermann [28]. Develin and Sturmfels [10] initiated the study of tropical convex polytopes and tropical discrete geometry. We recall the basic definitions. A set $C \subset \mathbb{R}^d$ is called **tropically convex** if for any two point $a, b \in C$, the **tropical line segment** $\{\lambda \odot a \oplus \mu \odot b | \lambda, \mu \in \mathbb{R}\}$ spanned by a and b is contained in C. We stress that the coefficients of a tropical convex combination $\lambda \odot a \oplus \mu \odot b$ are not required to be nonnegative. The **tropical convex hull** $\operatorname{tconv}(S)$ of a set $S \subseteq \mathbb{R}^d$ is defined as the

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[1]By definition, the operations in a semiring obey the same laws as the operations in a ring, including associativity and distributivity, except that one no longer requires the existence of a neutral element for addition (a zero) or of additive inverses. In the case of the tropical semiring, the underlying set $\mathbb{R}$ is sometimes augmented by the additively neutral element $+\infty$, but there are no additive inverses. Note that 0 is the multiplicatively neutral element, i.e., $0 \otimes a = a$ for all a, and that $-a$ is the tropical multiplicative inverse of an element a. The tropical semiring has the additional important property that tropical addition is **idempotent**, i.e., $a \oplus a = a$ for all a.

[2]If A is the adjacency matrix of a edge-weighted digraph then the (u, v)-entry of the k-th matrix power A^k (with tropical matrix multiplication) is the minimum weight of any path with k (possibly repeated) edges between vertices u and v. If all edge-weights are positive, then no edge is revisited and so the matrix $I \oplus A \oplus A^2 \oplus A^3 \oplus \ldots \oplus A^{n-1}$ encodes all minimum weight paths, where I is the matrix with 0's on the diagonal and ∞ elsewhere.

smallest tropically convex set containing S. It is easy to see [10, Proposition 4] that $\operatorname{tconv}(S)$ equals the set of all finite tropical convex combinations of points in S,

$$\operatorname{tconv}(S) = \{\bigoplus_{p\in X} \lambda_p \odot p \mid \lambda_p \in \mathbb{R}, X \subseteq S, X \text{ finite}\}.$$

By definition, a **tropical polytope** is the tropical convex hull of a finite set of points.

A tropically convex set C in $\mathbb{R}^{d+1}$ is closed under tropical multiplication by arbitrary scalars, i.e., along with any point $a \in C$, the convex set C contains the set $\mathbb{R} \odot a = \{\lambda \odot a \mid \lambda \in \mathbb{R}\} = a + \mathbb{R}\,\mathbf{1}$, which is just the classical line in direction $\mathbf{1} = (1, \ldots, 1)$ passing through a. Therefore, one identifies tropically convex sets in $\mathbb{R}^{d+1}$ with their projections in the d-dimensional **tropical projective space**

$$\mathbb{TP}^d = \{\mathbb{R} \odot a \mid a \in \mathbb{R}^{d+1}\} = \mathbb{R}^{d+1}/(\mathbb{R}\,\mathbf{1}).$$

Tropical polytopes have applications, e.g., to phylogenetic analysis in biology, see [10, 9]. They are also closely related to other well-studied objects in mathematics. For instance, a tropical convex polytope $P = \operatorname{tconv}(V)$ is a compact subset of $\mathbb{TP}^d$ that has a natural decomposition as a finite union of ordinary polytopes. This decomposition is called the **tropical complex** generated by V. Sturmfels and Develin showed that the combinatorial types of tropical complexes determined by an r-point set in $\mathbb{TP}^d$ are in bijection with the regular polydhedral subdivisions of the product of two simplices $\Delta^{r-1} \times \Delta^d$.

We remark that while a many classical results have nice analogues in tropical geometry (see the discussion below), in other cases, unexpected subtleties arise. For instance, there are three nonequivalent natural ways to define the rank of a tropical matrix [8]. Also, the definition of a face of a tropical convex polytopes is more subtle than in the classical case [16, 11].

Joswig [16] introduced **tropical halfspaces** and showed that tropical convex polytopes are precisely the bounded intersections of finitely many closed tropical halfspaces. By definition, a **tropical hyperplane** $\mathcal{H}_a \subset \mathbb{TP}^d$ defined by the tropical linear form $a = (a_0, \ldots, a_d) \in \mathbb{R}^{d+1}$ is the set of points $(x_0, \ldots, x_d) \in \mathbb{TP}^d$ such that the minimum $\bigoplus_{i=0}^d a_i \odot x_i = \min\{a_0 + x_0, \ldots, a_d + x_d\}$ is attained in at least two coordinates, see Figure 1. The point $-a$ is called the **apex** of $\mathcal{H}_a$. At the apex of a hyperplane, the minimum is simultaneously attained in every coordinate. Note that tropical hyperplanes only differ by translations. The complement of a hyperplane $\mathcal{H}$ in $\mathbb{TP}^d$ consists of $d + 1$ connected components, called the **open sectors** of $\mathcal{H}$. The topological closure of an open sector is called a **closed sector**. A **closed tropical halfspace** is the union of some subset of between 1 and $d-1$ closed sectors of a tropical hyperplane. The union of the complementary collection of closed sectors is the **complementary closed halfspace**.

The definition of tropical hyperplanes fits into the larger context of **tropical algebraic geometry**, which studies **tropical varieties**, i.e., subsets of $\mathbb{R}^d$ defined by (possibly higher order) tropical polynomials, i.e., finite tropical sums of tropical monomials $c \odot x_1^{\odot a_1} \odot \ldots \odot x_d^{\odot a_d}$. Each tropical monomial corresponds to a classical linear function $c + a_1x_1 + \ldots + a_dx_d$, and a tropical polynomial is the piecewise linear concave function obtained by taking the minimum of these linear functions. The corresponding tropical variety is defined as the set of points where the minimum is attained at least twice, which corresponds to the projection of the lower-dimensional faces of the lower envelope of the corresponding hyperplanes. For an introduction to tropical algebraic geometry and its applications we refer to [22, 12, 19].

Our contributions and related work. We provide several new results in tropical discrete geometry, by establishing the following tropical analogues of well-known classical results (see, e.g., [18] for the classical versions and further background).

Theorem 1 (**Tropical Polytope-Polytope Separation**). *If P and Q are two disjoint tropical polytopes in $\mathbb{TP}^d$ then there is a tropical hyperplane such that P and Q are contained in the interiors of complementary closed halfspaces.*

This theorem generalizes earlier tropical Farkas Lemma type results concerning point-polytope separation [10, 16]. See also [7] for further tropical Hahn-Banach-type results concerning the separation of a point from a convex set.

Theorem 2 (**Tropical Centerpoint Theorem**). *Let S be a set of n points in $\mathbb{TP}^d$. Then there exists a **tropical centerpoint** for S, i.e., a point $c \in \mathbb{TP}^d$ such that any closed tropical halfspace containing c contains at least $\frac{n}{d+1}$ points of S. Given S, a tropical centerpoint can be computed in time $O(n^4)$ independent of d.*

We briefly contrast this to the situation in classical geometry. Jadhav and Mukhopadhyay [15] gave a linear-time algorithm algorithm for computing centerpoints in the plane, Naor and Sharir [21] proposed an $O(n^2 \operatorname{polylog}(n))$ algorithm for 3 dimensions, and Chan [4] described a randomized $O(n \log n + n^{d-1})$ algorithm for any fixed dimension d. Teng [26] showed that deciding whether a given point c is a (classical) centerpoint (or an r-Tverberg point, see below) of a point set in $\mathbb{R}^d$ is coNP-complete if d is part of the input. An algorithm for computing approximate centerpoints in variable dimension was proposed in [5].

Theorem 3 (**Tropical Tverberg Theorem**). *Let $r, d \geq 1$ and let S be a set of $n \geq (d+1)(r-1)+1$ points in $\mathbb{TP}^d$. Then there exists a **tropical Tverberg partition** into r parts, i.e., r pairwise disjoint subsets $S_1, \ldots, S_r \subseteq S$ such that $\bigcap_{i=1}^r \operatorname{conv}(P_i) \neq \emptyset$. Any point in this intersection is called a **tropical r-Tverberg point** of S. Given S as above, a tropical r-Tverberg partition can be computed in time $O(nd)$.*

Again, in classical geometry there are no efficient algorithms solving this problem in variable dimension. See also [1] for more background on the algorithmic aspects of centerpoints and Tverberg points. Both in classical and tropical setting, every Tverberg point is a centerpoint. We actually show that for tropical point sets in general position, the reverse implication also holds.

From the tropical version of **Radon's Lemma** (which is just Tverberg's Theorem for the case $r = 2$) one also easily derives a tropical analogue of **Helly's Theorem**: a family of $n \geq d+1$ tropically convex sets in $\mathbb{TP}^d$ has a nonempty intersection iff any $(d+1)$ of the sets have a nonempty intersection.

Theorem 4 (**Tropical Colorful Carathéodory Theorem**). *Consider $d+1$ finite point sets $M_1, \ldots, M_{d+1}$ in $\mathbb{TP}^d$ such that $\mathbf{0} \in \operatorname{tconv}(M_i)$ for every i. Then there exists "colorful" a $(d+1)$-point set $S \subseteq M_1 \cup \ldots \cup M_{d+1}$ with $|M_i \cap S| = 1$ for each i such that $\mathbf{0} \in \operatorname{tconv}(S)$.*

This extends the basic ("colorless") tropical version of Carathéodory's theorem proved in [10]. We also give an elementary proof of the following result; we consider this of particular interest, since the proof of the classical version uses methods from equivariant topology [29, 17].

Theorem 5 (**Tropical Colored Tverberg Theorem**). *For any $k, d \geq 2$, any point set P of $n \geq k(d+1)$ points in general position in $\mathbb{TP}^d$ that comes partitioned into $d+1$ color classes $A_0, \ldots, A_d$ with k points each, there exist k pairwise disjoint sets $P_1, \ldots, P_k \subseteq P$ such that each P_i contains exactly one point of each color A_j, $j = 0, \ldots, d$ (i.e. the P_i are "rainbow"), and $\bigcap_{i=1}^k \operatorname{tconv}(P_i) \neq \emptyset$.*

Tropical Halfspace Arrangements and k-Sets. In Section 4, we count the number of regions in a tropical hyperplane arrangement, a problem very related to tropical polytopes [10]. We will later use this to bound the number of tropical k-sets. A **k-set** of an n-point set P is a subset of k points that can be separated from the other points by a halfspace. For classical k-sets, there remains a big gap between the proven lower and upper bound for the number of k-sets [18, Chapter 11] or [27]. For tropical k-sets, we prove a tight bound of $\Theta(n^{d-1})$ for the number of k sets of an n-point sets in fixed dimension d (assuming $k \in \Theta(n)$ and $n - k \in \Theta(n)$).

2 Basics of Tropical Geometry

Recall that a point in $\mathbb{TP}^d$ corresponds to a classical line in direction $\mathbf{1} = (1, \ldots, 1)$ in $\mathbb{R}^{d+1}$. There are two standard types of coordinates used to represent a such point. The first is in **canonical coordinates**, for which we take the unique point on the line such that all coordinates are nonnegative and at least one coordinate is zero. The second is in **normalized coordinates**, where we choose the unique point on the line whose first coordinate is zero.

For a point $x = (x_0, \ldots, x_d) \in \mathbb{TP}^d$ in canonical coordinates, let $\|x\| = \max_{i=0}^d x_i$. For a point $x = (x_0, \ldots, x_d) \in \mathbb{TP}^d$ in arbitrary coordinates, this means that $\|x\| = \max\{x_i - x_j | i \neq j\}$. It is easy to see that $d(x, y) := \|x - y\|$ defines a metric on $\mathbb{TP}^d$, see [16, Lemma 2.1].

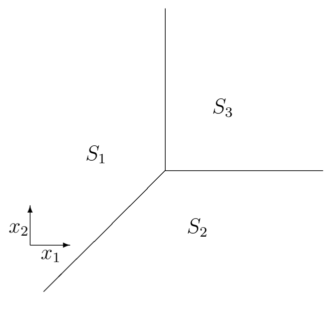

Figure 1: A tropical hyperplane in $\mathbb{TP}^2$ (in normalized coordinates, $x_0 = 0$).

General position. The projection from $\mathbb{R}^{d+1}$ onto any coordinate hyperplane is a homomorphism of tropical semimodules [10, Theorem 2] and therefore induces a map $\mathbb{TP}^d \to \mathbb{TP}^{d-1}$. Any map obtained by an iteration of this process is called **projection** onto a **tropical coordinate subspace**. A point set $S \subseteq \mathbb{TP}^d$ is said to be in **tropically general position** (see [10, Proposition 24]) if no $k+1$ of the points have a projection onto a k-dimensional coordinate subspace $\mathbb{TP}^k$ in which all projected points lie in a common hyperplane, $1 \leq k \leq d$. A tropical hyperplane arrangement is in general position if the apices of the hyperplanes are in tropical general position.

Sectors and duality for tropical hyperplanes. Recall that $\mathcal{H}_a$ denotes the tropical hyperplane with apex at the point $-a$, see Figure 1. For the hyperplane $\mathcal{H}_0$ with apex at the origin, its **open sectors** are the sets $S_0, \ldots, S_d$, where $S_i = \{(x_0, \ldots, x_d) \mid x_i < x_j \text{ for all } j \neq i\}$. Similarly, the closed sectors are denoted by $\bar{S}_0, \ldots, \bar{S}_d$. For a general hyperplane $\mathcal{H}_a$, its i-th open sector is $-a + S_i$. Note that a point lies on a tropical hyperplane if it lies in at least two closed sectors. A point $p \in \mathbb{TP}^d$ and the hyperplane $\mathcal{H}_p$ are called **duals** of each other. This duality preserves incidences and containment in labeled sectors:

Observation 6. *For $a, p \in \mathbb{TP}^d$ and $0 \leq i \leq d$, p lies in the (open/closed) sector i of $\mathcal{H}_a$ iff a lies in the (open/closed) sector i of $\mathcal{H}_p$. Moreover, $p \in \mathcal{H}_a$ iff $\in \mathcal{H}_p$.*

2.1 Systems of Tropical Linear Equations and Cramer's Rule

One technical issue that arises in tropical geometry is that the definition of tropical linear subspaces is somewhat subtle and that, in particular, the set-theoretic intersection of finitely many tropical hyperplanes need not be a tropical linear subspace, see the discussion in [22]. To circumvent this a better-behaved definition of an intersection using a tropical variant of Cramer's rule was suggested in [22] (in general position, both definitions agree). Cramer's rule will be an important tool in our proof of the Centerpoint Theorem 2. We recall some definitions.

Definition 7 ([22])**.** *For a $k \times k$-matrix $C = (c_{i,j})$, the* ***tropical determinant*** *is*

$$\det_{trop}(C) = \bigoplus_{\pi \in S_k} (c_{1,\pi(1)} \odot \ldots, \odot c_{k,\pi(k)}),$$

where S_k is the group of permutations on $[k]$. A matrix is ***tropically singular*** *if the tropical determinant is attained at more than one permutation.*

Lemma 8 (Geometric interpretation of a singular matrix, [22, Lemma 5.1])**.** *A $k \times k$-matrix A is tropically singular iff the k points whose coordinates are the row (or column) vectors of A lie on a tropical hyperplane in $\mathbb{TP}^{k-1}$.*

Lemma 9 (Alternative characterization of general position)**.** *A n-point set S in $\mathbb{TP}^d$ is in general position if no $k \times k$-submatrix of the $n \times (d+1)$-matrix $(s_{i,j})$ whose entries are the coordinates of the points is tropically singular for any $2 \leq k \leq n$.*

Proof. The coordinates of a projection of k points onto a k-dimensional coordinate subspace is a $k \times k$-submatrix of the matrix $(s_{i,j})$ and vice versa. The result then follows from the previous Lemma 8. ☐

Definition 10 ([22, Section 5])**.** *Consider $k \leq d$ hyperplanes $\mathcal{H}_{a_1}, \ldots, \mathcal{H}_{a_k}$ in $\mathbb{TP}^d$. Let $A = (a_{i,j})$ be $k \times (d+1)$-matrix whose i-th row contains the apex of the hyperplane $\mathcal{H}_{a_i}$. For any k-subset I of $[0..d]$ let A_I be the $k \times k$-submatrix of A consisting of the columns I, and define $w_I := \det_{trop}(A_I)$ to be its tropical determinant.*

For any $(d-k+2)$-subset J of $[0..d]$ and its complement J^C, we define $G_J := \bigoplus_{j \in J} w_{J^c \cup j} \odot x_j = \min_{j \in J}\{w_{J^c \cup j} \odot x_j\}$, and let $\mathcal{T}(G_J)$ be the set of points x for which is attained at least twice. Then

$$\bigcap_{J \subseteq [0..d], |J| = d-k+2} \mathcal{T}(G_J) \qquad \subset \mathbb{TP}^d, \tag{1}$$

is called the ***Cramer intersection*** *of the hyperplanes $\mathcal{H}_{a_i}$.*

The following theorem summarizes the most important properties of the Cramer intersection.

Theorem 11 ([22, Theorem 5.3])**.** *The Cramer intersection is a region of codimension k in $\mathbb{TP}^d$. It is always contained in the intersection of the hyperplanes. For hyperplanes, the two intersections are equal if and only if none of the $k \times k$-submatrices A_I of A is tropically singular.*

Lemma 12. *For $k = d$, the Cramer intersection consists of just one point, and it is continuous in the coordinates of the points a_i.*

Proof. For this, we have to look at the G_J. For $k = d$, J is a 2-element subset of $[0..d]$. Let $J = \{j_1, j_2\}$. Then $G_J = w_{J^c \cup j_1} \odot x_{j_1} \oplus w_{J^c \cup j_2} \odot x_{j_2}$. This means that for $\mathcal{T}(G_J)$, $w_{J^c \cup j_1} \odot x_{j_1} = w_{J^c \cup j_2} \odot x_{j_2} \Leftrightarrow x_{j_1} - x_{j_2} = w_{J^c \cup j_2} - w_{J^c \cup j_1}$. As $w_{J^c \cup j_2}$ and $w_{J^c \cup j_1}$ are tropical determinants, and tropical determinants are continuous in the coordinates of the points, the Cramer intersection is continuous. □

Note that the coordinates of the point in the Cramer intersection of d hyperplanes $\mathcal{H}_{a_1}, \ldots, \mathcal{H}_{a_d}$ are $(w_{A|0}, w_{A|1}, \ldots, w_{A|d})$, where $w_{A|i}$ is the tropical determinant of the $d \times d$-matrix that we get if we delete the i-th column of $A = (a_{i,j})$. This holds, as in the proof above, $w_{J^c \cup j_2} = w_{A|j_1}$ and $w_{J^c \cup j_1} = w_{A|j_2}$. So, the problem of computing the Cramer intersection is equivalent to computing the tropical determinants of $d+1$ different $d \times d$-matrices. The tropical determinant of a $d \times d$-matrix $C = (c_{i,j})$ can be computed by solving an assignment problem. For this, consider the weighted complete bipartite graph with d vertices on each side, where the weight on the edge between the i-th vertex on the left and the j-th vertex on the right side is $c_{i,j}$. Then the tropical determinant is the same as the value of a minimum weight matching on the graph. With the Hungarian algorithm, such a matching can be computed in $O(d^3)$ (see for instance [24]).

Corollary 13. *The Cramer intersection of d tropical hyperplanes in $\mathbb{TP}^d$ can be computed in time $O(d^4)$.*

In the proof of the tropical Centerpoint Theorem 2, we will use the following fact:

Lemma 14. *If the same row r appears k times in a $d \times (d+1)$ matrix A, then the Cramer intersection x has at least $k+1$ coordinates that minimize $x + r$.*

Proof. We augment the matrix A to a square $(d+1) \times (d+1)$-matrix A' by adding another copy of the row r on top of it, and assume w.l.g. that the first $k+1$ rows of A' are the copies of r. Let π be a permutation where $\det_{trop}(A')$ attains its minimum. This implies

$$\begin{aligned} \det_{trop}(A') &= r_{\pi(1)} + \det_{trop}(A|_{\pi(1)}) = r_{\pi(2)} + \det_{trop}(A|_{\pi(2)}) = \cdots = r_{\pi(k+1)} + \det_{trop}(A|_{\pi(k+1)}) \\ &= r_{\pi(1)} + x_{\pi(1)} + \lambda = r_{\pi(2)} + x_{\pi(2)} + \lambda = \cdots = r_{\pi(k+1)} + x_{\pi(k+1)} + \lambda \end{aligned}$$

for some $\lambda \in \mathbb{R}$ (see also remark after Lemma 12), which proves our claim. □

2.2 Facts from Tropical Convexity

We recall a few basic facts about tropical convexity (see [10, 16]) that will be useful for what follows. The first is a simple sufficient and necessary condition for a point to lie in the convex hull of a point set.

Proposition 15. *[16, Proposition 2.9] Let $X = \{x_1, \ldots, x_n\} \in \mathbb{TP}^d$. Then $p \in \operatorname{tconv}(X)$ iff each closed sector $i \in [0..d]$ of $\mathcal{H}_{-p}$ contains at least one point in X.*

This means that $p \in \operatorname{tconv}(X)$ iff for each coordinate $j \in [0..d]$, there exists an x_i such that $(x_i - p)_j = 0$ for $x_i - p$ in canonical coordinates.

Theorem 16. *The following sets are tropically convex: tropical hyperplanes [10, Proposition 6], closed tropical halfspaces [16, Proposition 2.17] and open tropical halfspaces [16, Corollary 2.18] (in particular, this applies to open or closed sectors), the boundaries of tropical halfspaces [16, Corollary 2.18], intersections of two tropically convex sets [10, Theorem 2].*

Lemma 17 ([16, Lemma 2.16])**.** *Let $a + \bar{S}_k \in \mathbb{TP}^{d-1}$ be a closed sector for some $k \in [0..d]$, and $b \in a + \bar{S}_k$. Then the "parallel" sector $b + \bar{S}_k$ is contained in $a + \bar{S}_k$.*

3 Tropical Versions of Classical Theorems in Convexity

3.1 Separating Two Convex Polytopes

The classical Farkas' Lemma states that a point p is either contained in a polytope or is separable from it by a halfspace. In the tropical case, the analogous statement was shown in [16, Theorem 2.19]. Our separation theorem 1 is a stronger statement. Instead of just a point and a polytope, we separate two polytopes.

In classical the classical case, the proof idea very simple: For any pair $p \in P$ and $q \in Q$ that minimize $\|p - q\|$, any hyperplane orthogonal

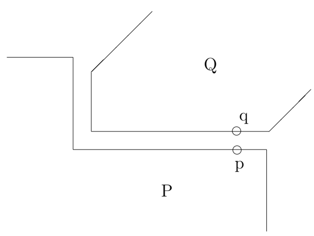

Figure 2: A pair of nearest points p and q.

to the segment pq and passing through its midpoint, say, is easily seen
to separate P from Q. In tropical geometry, however, an arbitrary pair of nearest points is in general not sufficient to determine a halfspace that separates the two sets (see Figure 2). In order to create a stronger nearest notion of nearest point pairs, we use the tropical nearest point map as defined in [10]:

Definition 18. *For a polytope $P = \mathrm{tconv}(V)$ in $\mathbb{TP}^d$, where $V = (v_1, v_2, \ldots, v_n)$, the **tropical nearest point map** on P $\pi_P : \mathbb{TP}^d \to P$ is defined as*

$$\pi_P(x) = \lambda_1 \odot v_1 \oplus \lambda_2 \odot v_2 \oplus \ldots \oplus \lambda_n \odot v_n,$$

where $\lambda_i = \min\{\lambda \in \mathbb{R} : \lambda \odot v_i \oplus x = x\}$.

We will now show that there exist a pair of points $p \in P$ and $q \in Q$ which map to each other under the respective nearest point maps.

Lemma 19. *For any polytope P, π_P is weakly contracting; i.e. $\|\pi_P(x) - \pi_P(y)\| \leq \|x - y\|$ $\forall x, y$.*

Proof. By translation invariance, we may assume that $x = (0, 0, \ldots, 0)$ and $y = (y_0, y_1, \ldots, y_{d-1}, 0)$, where $y_0 \geq y_1 \geq \ldots \geq y_{d-1} \geq 0$ (after a possible re-ordering of the coordinates). Let $V = (v_1, v_2, \ldots, v_n)$ be the vertices of our polytope $P = \mathrm{tconv}(V)$, where the $v_i = (v_{i,0}, \ldots v_{i,d})$ are in canonical coordinates. Then $\pi_P(x) = \bigoplus_{i=1}^n v_i$, since $\min\{\lambda \in \mathbb{R} : \lambda \odot v_i \oplus x = x\} = 0$. Furthermore, $\pi_P(y) = \bigoplus_{i=1}^n \lambda_i \odot v_i$ for $\lambda_i := \min\{\lambda \in \mathbb{R} : \lambda \odot v_i \oplus y = y\}$. Since every v_i contains a zero coordinate, and y is non-negative, $\lambda_i \geq 0$ for every i. Further, every v_i is nonnegative, so $y_1 \odot v_i$ is greater or equal than y in every coordinate, implying $\lambda_i \leq y_1$. So, for $0 \leq j \leq d$,

$$\pi_P(x)_j = \bigoplus_{i=1}^n v_{i,j} \leq \bigoplus_{i=1}^n \lambda_i \odot v_{i,j} = \pi_P(y)_j \leq \bigoplus_{i=1}^n y_0 \odot v_{i,j} = y_0 + \pi_P(x)_j.$$

$$\Rightarrow \quad 0 \leq \pi_P(y)_j - \pi_P(x)_j \leq y_0 \quad \Rightarrow \quad \|\pi_P(x) - \pi_P(y)\| \leq y_0 = \|x - y\|.$$ □

Lemma 20. *For any two tropical polytopes P and Q, there exists a pair of points $x \in P$ and $y \in Q$ such that $y = \pi_Q(x)$ and $x = \pi_P(y)$.*

Proof. We use that $P, Q \in \mathbb{TP}^d$ are compact sets. Let B be a closed (classical) ball in $\mathbb{TP}^d$ that contains P. We define a function $f : B \to P \subseteq B$ as $f(x) := \pi_P(\pi_Q(x))$. Due to Lemma 19, π_Q and π_P are both weakly contracting, so f is also weakly contracting. This implies that f is continuous. Consequently, due to the Brouwer fixed point theorem [20], there exists a point x such that $x = f(x)$. Since f maps points to P, we must have $x \in P$, and for $y = \pi_Q(x)$, $\pi_P(y) = \pi_P(\pi_Q(x)) = f(x) = x$. □

Proof of the Separation Theorem 1. Let $x \in P$ and $y \in Q$ such that $y = \pi_Q(x)$ and $x = \pi_P(y)$, according to Lemma 20. We again assume w.l.g. that $x = (0, \ldots, 0)$ and $y = (y_0, \ldots y_{d-1}, 0)$, where $y_0 \geq y_1 \geq \ldots \geq y_{d-1} \geq 0$. Now we examine the position of Q relatively to y. Let $Q = \mathrm{tconv}(V) = \mathrm{tconv}(v_1, \ldots, v_k)$ where the $v_i = (v_{i,0}, v_{i,1}, \ldots, v_{i,d})$ are again in canonical coordinates. For $0 \leq j \leq d$,

$$y_j = \pi_Q(x)_j = \bigoplus_{i=1}^k v_{i,j} \leq v_{i,j} \quad \forall\ 1 \leq i \leq k,$$

so $v_i - y$ is non-negative for all i. Since y also is non-negative, $v_{i,j} = 0$ always implies that $y_j = 0$ and $v_{i,j} - y_j = 0$. Consequently, any minimal coordinate of any v_i is also a minimal coordinate of y and of $v_i - y$. So, each point v_i lies in $\bigcup_{j|y_j=0} y + \bar{S}_j$. Because this is a closed halfspace, and hence convex by Theorem 16, Q is contained in this halfspace.

Applying the same argument as above, for P instead of Q, we deduce that P lies in $\bigcup_{k|y_k=y_0} x + \bar{S}_k$.

Let the i-th coordinate be the smallest nonzero coordinate of y, and for some fixed $0 < \lambda < y_i$, consider the point $s_\lambda := \lambda \odot x \oplus y = (\lambda, \ldots, \lambda, 0, \ldots, 0)$. We claim that P is contained in the open halfspace with apex in s_λ and sectors 0 to i, and Q is contained in its open complement.

Let $p \in P$. Then as $x = 0$, we have shown that for p, one of the coordinates k, for which $y_k = y_0$, is minimal. For $p - s_\lambda$, this means that the minimal coordinates are all in the first i positions. This implies that p is in the desired halfspace.

Let $q \in Q$. For at least one minimal coordinate j of $q - y$ we have $y_j = 0$ and consequently $(s_\lambda)_j = 0$. Since $y - s_\lambda$ is nonnegative and only zero in coordinates j where $y_j = 0$, $(q - y) + (y - s_\lambda) = q - s_\lambda$ is only minimal in coordinates where $y_j = 0$, as required. □

3.2 The Centerpoint Theorem and Tverberg's Theorem

Recall that for a set P of n points in $\mathbb{TP}^d$, a point $c \in \mathbb{TP}^d$ is called a **tropical centerpoint** of P if each tropical closed halfspace containing c contains at least $\frac{n}{d+1}$ points of P. A hyperplane with apex on a tropical centerpoint contains at least $\frac{n}{d+1}$ points in each closed sector. Further, an apex of a hyperplane that contains at least $\frac{n}{d+1}$ points in each closed sector always is a centerpoint, as by Lemma 17, any closed tropical halfspace containing the apex also contains at least one sector of the hyperplane.

Proof of tropical Centerpoint Theorem 2. Let P be a set of n points in $\mathbb{TP}^d$. We want to show that a tropical centeropoint for P exists and can be computed in time $O(n^4)$. We may assume that $n = k(d+1)+1$ for a $k \in \mathbb{N}$ (since for $k(d+1)+1 < n < (k+1)(d+1)+1$, every tropical centerpoint of a $(k(d+1)+1)$-subset of P is also a centerpoint of P). We interpret P as a $(d+1) \times n$ matrix $(p_{i,j})$, with the j-th column corresponding to the j-th point. Let $r_i \in \mathbb{R}^n$ be the i-th row vector of the matrix $P = (p_{i,j})$. In $\mathbb{R}^n = \mathbb{R}^{k(d+1)+1}$, we compute the Cramer intersection of k copies of each of of the hyperplanes $\mathcal{H}_{r_i}$, thus the intersection of $k(d+1)$ hyperplanes in total. By Lemma 14, the solution $x \in \mathbb{R}^n$ has the additional property that the minimum of *each* $x + r_i$ is attained at least $k+1$ times. We define $q \in \mathbb{R}^{d+1}$ to contain exactly those minima, i.e. $(x+r_i)_j - q_i = x_j + p_{i,j} - q_i \geq 0 \; \forall j \in [n]$, and $= 0$ for at least $k+1$ many indices j, for any fixed $i \in [0..d]$. But now we are done, because if we interpret this in $\mathbb{TP}^d$, this means nothing else than for any $i \in [0..d]$, $p_j - q$ attains its minimum in coordinate i (the value of the minimum is $-x_j$), for at least $k+1$ many points p_j, or in other words that the hyperplane with apex q contains $k+1$ points in every closed sector i. This is equivalent to q being a centerpoint by our above remark. We conclude the proof by observing that algorithmically, the intersection of $k(d+1)$ many tropical hyperplanes can be computed in time $O((k(d+1))^4) = O(n^4)$ as stated in Theorem 11. □

Our above proof can also be interpreted in view of the following isomorphism; instead of finding the centerpoint q directly, our algorithm finds $-x$, which then maps back to q under the isomorphism:

Theorem 21 ([10, Theorem 23])**.** *Given any matrix $M \in \mathbb{R}^{n\times d}$, the tropical convex hull of its column vectors is isomorphic to the tropical convex hull of its row vectors. The isomorphism is obtained by restricting the piecewise linear maps $\mathbb{R}^d \to \mathbb{R}^n$ $z \mapsto M \odot (-z)$, and $\mathbb{R}^n \to \mathbb{R}^d$ $y \mapsto (-y) \odot M$.*

Next, we want to prove the tropical Tverberg Theorem 3. We restrict ourselves to the case of point sets in general position (the degenerate case follows by a simple compactness argument). Thus, let P be a set of at least $(d+1)(r-1)+1$ points in general position in $\mathbb{TP}^d$, where $r, d \geq 1$. Recall that a partition $P = P_1 \dot\cup \ldots \dot\cup P_r$ with $\bigcap_{i=1}^r \operatorname{tconv}(P_i) \neq \emptyset$ is called an r-Tverberg partition of P, and any point in the intersection is called an r-Tverberg point. As in the classical case, it is easy to see that

Lemma 22. *Every tropical Tverberg point of P is a tropical centerpoint.*

We will show that in the tropical case, the reverse implication also holds and that, given a centerpoint (which we can compute in time $O(n^4)$ by Theorem 2), we can compute a corresponding Tverberg partition in time $O(nd)$.

Proof of Theorem 3. Let us look at a hyperplane $\mathcal{H}$ with apex at a tropical centerpoint of P. Each closed sector of $\mathcal{H}$ contains at least r points. We look at the following bipartite graph $G = (V, E)$ given by the position of the points relative to this hyperplane: $V := [0..d] \uplus P$, $E := \{\{i, j\} \mid i \in [0..d], j \in P, p \text{ lies in the closed sector } i \text{ of } \mathcal{H}\}$. What we need to show is that we can color the vertices P with r colors so that every vertex in $[0..d]$ is adjacent to r different colors. If we partition the points according to their color, then each partition has a point in each closed sector of $\mathcal{H}$. According to Proposition 15, this means that the apex of $\mathcal{H}$ is in the convex hull of each such partition, proving the theorem.

As each closed sector of $\mathcal{H}$ contains at least r points, $|E| \geq (d+1)r$. We now prove that G is cycle-free, implying that it is a tree, since $|V| = r(d+1)+1$. Assume for sake of contradiction that G contains a cycle, and assume w.l.g. that the shortest cycle is $0, p_0, 1, p_1, 2, \ldots, k, p_k, 0$. We look at the projections $p'_0, \ldots, p'_k$ of the first $k+1$ points onto the subspace spanned by the first $k+1$ coordinates. As coordinates 0 to k are unaffected by this projection, for $0 \leq i \leq k-1$, the points p'_i lie in both sectors i and $i+1$, and p'_k lies in both sectors 0 and k. So, each of these points lies on the projection of $\mathcal{H}$, which is a tropical hyperplane. This is a contradiction to the assumption that the points are in general position.

What is left is coloring the graph according to the above conditions. This can be done greedily: Starting at a vertex in $[0..d]$, we color all adjacent edges with different colors. We then traverse the tree, always maintain that all vertices adjacent to a vertex in $[0..d]$ are colored differently. □

Corollary 23. *Every tropical centerpoint of P is also a tropical Tverberg point. Given a centerpoint a corresponding tropical r-Tverberg partition can be computed in $O(nd)$ time.*

Proof. Given a tropical r-Tverberg point p, we need $O(n(d+1))$ time to determine in which sector of $\mathcal{H}_{-p}$ each point in P lies. After that, partitioning the points can be done by graph-traversal, in time $O(|V|) = O(r(d+1)+1) = O(n+d)$. □

This is in contrast to classical geometry, where there exist point configurations for any $n = (r-1)(d+1)+1$ points in general position in $\mathbb{R}^d$ where $d \geq 3$ and $r \geq 3$ such that some centerpoints are not Tverberg points. Moreover, there is no known polynomial time algorithm for computing classical Tverberg points or Tverberg partitions in variable dimension [1].

The special case $r = 2$ yields:

Corollary 24 (Tropical Radon Lemma). *Every set $P \subset \mathbb{TP}^d$ of $d+2$ or more points can be partitioned into two disjoint sets P_1 and P_2 such that $\operatorname{tconv}(P_1) \cap \operatorname{tconv}(P_2) \neq \emptyset$.*

Completely analogously as in the classical case (see e.g. [18, Section 1.3]), one can use Radon's Lemma and induction to prove:

Theorem 25 (Tropical Helly Theorem). *Let $C_1, C_2, \ldots, C_n$ be tropically convex sets in $\mathbb{TP}^d$, $n \geq d+1$. Suppose that the intersection of every $d+1$ of these sets is nonempty. Then the intersection of all the C_i is nonempty.*

We remark that, like its classical cousin, the tropical Helly Theorem is sharp, in the following sense:

Proposition 26. *There exists a set of $d+1$ tropically convex sets $C_0, C_1, \ldots, C_d$ in $\mathbb{TP}^d$ such that the intersection of any d of these is nonempty, but $\bigcap_{i=0}^{d} C_d = \emptyset$.*

Proof. Consider the sets $C_i := \{(x_0, \ldots, x_d) \mid x_i \geq x_j \text{ for } j \in [0..d]\} \setminus \{\mathbf{0}\}$. These are tropically convex, since for two points $x_1, x_2 \in C_i$ and $\lambda, \mu \in \mathbb{R}$ it holds that

$$(\lambda \odot x_1 \oplus \mu \odot x_2)_i = (\lambda \odot x_1)_i \oplus (\mu \odot x_2)_i \geq (\lambda \odot x_1)_j \oplus (\mu \odot x_2)_j = (\lambda \odot x_1 \oplus \mu \odot x_2)_j$$

for any $j \in [0..d]$. Each set C_i contains d points of $S = \{-e_j | j \in [0..d]\}$, so any intersection of d of these sets contains at least one point of S. Finally, for a point in $\bigcap_{i=0}^{d} C_d$, every coordinate needs to be maximal. However, the only point with this property is the origin, which does not lie in any set C_i. □

We can use Radon's Lemma in the framework of VC-dimension. A **range space** is a tuple (X, R) of a set X (called the **points**) and a family of subsets $R \subseteq 2^X$ (called the **ranges**). A set $S \subseteq X$ can be **shattered by** R if $|\{r \cap S | r \in R\}| = 2^{|S|}$. The **VC-dimension** of a range space VC-dim(X, R) is the maximal cardinality of a subset of X that can be shattered by R.

Let $\mathbf{H}^d$ be the family of halfspaces in $\mathbb{TP}^d$, and let $\mathbf{S}^d \subset \mathbf{H}^d$ be the family of single sectors in $\mathbb{TP}^d$., so VC-dim$(\mathbb{TP}^d, \mathbf{H}^d) \geq$ VC-dim$(\mathbb{TP}^d, \mathbf{S}^d)$. Tropical Radon's Lemma states that for $d+1$ points in $\mathbb{TP}^d$, there exists a partition into two sets with intersecting convex hull, which are therefore not separable by any halfspace. This shows that VC-dim$(\mathbb{TP}^d, \mathbf{H}^d) \leq d$. On the other hand it is obvious that VC-dim$(\mathbb{TP}^d, \mathbf{H}^d) = d$. For this, consider a set of d points in $\mathbb{TP}^d$, one in each open sector S_i. Then these points are shattered by halfspaces with apex at $\mathbf{0}$. The following Lemma strengthens this by showing that also VC-dim$(\mathbb{TP}^d, \mathbf{S}^d) = d$.

Lemma 27. *There exists a set of d points in $\mathbb{TP}^d$ that can be shattered by single sectors.*

Proof. We claim that $P = (p_1, \ldots, p_d)$, $p_i := -\mathbf{e}_i$ is such a set. Let $0 < c_1 < c_2 < 1$ and let $Q = (q_1, \ldots, q_k)$ be any nonempty subset of P. We choose the apex a of a hyperplane as follows: $a_i := 1 + c_1$ if $p_i = q_1$, $a_i := c_2$ if $p_i \notin Q$, and $a_i := 0$ otherwise. It is easy to verify that for i such that $p_i = q_1$, it holds that $p_j \in a + S_i$ iff $p_j \in Q$. □

3.3 Colorful Theorems

Proof of Theorem 4. Applying the characterization of convexity from Proposition 15, we know that each closed sector of the tropical hyperplane with apex at 0 contains at least one point of each M_i. So for each closed sector i, we select one point of color M_i to be included in S. This already is sufficient, since again Proposition 15 will imply that $0 \in \operatorname{tconv}(S)$. □

An elementary proof of the tropical colored Tverberg Theorem 5 is given in Appendix A, and is interestingly much easier than the proof of its classical counterpart which requires topological methods [29, 17].

4 Counting Problems

4.1 Tropical Hyperplane Arrangements

We consider the complexity of tropical hyperplane arrangements, following the work of Ardila, Develin and Sturmfels [2, 10]. A tropical hyperplane arrangement is the polyhedral decomposition of $\mathbb{TP}^d$ given by a set of hyperplanes. If two points lie in the same region of a given hyperplane arrangement, we say they have the same type:

Definition 28. *Given a tropical hyperplane arrangement $\mathcal{H}_1, \ldots, \mathcal{H}_n$ in $\mathbb{TP}^d$, the **type** of a point x is the n-tuple $(A_1, \ldots, A_n)$, where $A_i \subseteq [0..d]$ is the set of closed sectors of the hyperplane H_i that contain x.*

Definition 29. *A **tope** is a type $A = (A_1, \ldots, A_n)$ where each A_i is a singleton. We also use the notation $A(a_1, \ldots, a_n)$ for topes, if $A_i = \{a_i\}$.*

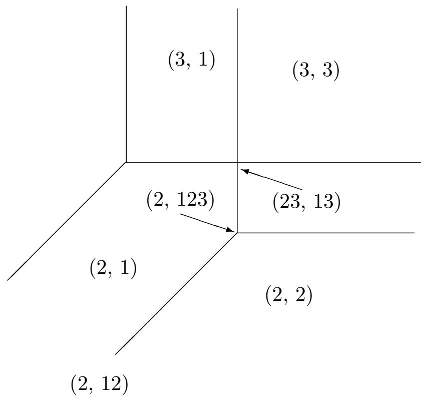

Figure 3: Some types of an arrangement of 2 hyperplanes in $\mathbb{TP}^2$.

Due to the following result, the bounded regions in a hyperplane arrangement $\mathcal{H}_{p_1}, \ldots, \mathcal{H}_{p_n}$ correspond to a natural decomposition of the polytope $\mathrm{tconv}(p_1, \ldots, p_n)$ as a finite union of ordinary polytopes, which is called the *tropical complex* generated by $P = (p_1, \ldots, p_n)$:

Theorem 30. *[10, Theorem 15] The convex hull of a finite point set P is the union of the bounded regions of the polyhedral decomposition of $\mathbb{TP}^d$ given by putting an inverted hyperplane at each point of P.*

Also, [10] showed that the types of this hyperplane arrangement determined by P are in bijection with the regular triangulations of the product of two simplices $\Delta^{n-1} \times \Delta^d$, of which the complexity is well understood:

Corollary 31. *[10, Corollary 25] All tropical polytopes spanned by n points in general position in $\mathbb{TP}^d$ have the same f-vector. Specifically, the number of faces of dimension k is equal to the multinomial coefficient $\binom{n+d-k-1}{n-k-1,\, d-k,\, k}$.*

Observation 32. *A type corresponds to a bounded region iff every direction $i \in [0..d]$ occurs in at least one coordinate. A type is unbounded in direction i iff i does not occur in any coordinate.*

Proof. We refer to [10, Corollary 12]. Note that [10] uses a different notation for a type $S = (S_1, \ldots, S_d)$, with S_i denoting the set of hyperplanes for which a given point lies in sector i. ☐

In the following we use the geometric equivalent of a projection operation by [2], such that for each unbounded region, there exists exactly one such projection for which the region becomes bounded.

Proposition 33. *We define the **geometric contraction** of a tropical hyperplane arrangement in dimensions S to be its projection onto the coordinate subspace $[0..d] \setminus S$. Then a geometric contraction of a tropical hyperplane arrangement in d dimensions is a tropical hyperplane arrangement in $d - |S|$ dimensions. If a tropical hyperplane arrangement is in general position, then all its geometric contractions also are in general position. If M is the set of types in a tropical hyperplane arrangement, then the set of types in its geometric contraction in dimensions S is $M_{/S}$, which consists of all types of M which do not contain any element of S in any coordinate.*

Proof. The first two claims are obvious, so we only need to prove the last claim.

Let A be a type of M which does not contain any element of S in any coordinate. Then for a point p with type A, we have that

$$S \cap \{\mathrm{argmin}(p+q) \mid \mathcal{H}_q \text{ is a hyperplane in the arrangement}\} = \emptyset \,.$$

This implies that for each hyperplane $\mathcal{H}_q$, $\mathrm{argmin}(p+q)$ is preserved when projecting onto the coordinate subspace, so the type A is preserved. Therefore, every type of $M_{/S}$ still exists in the geometric contraction in directions S. On the other hand for an arbitrary point p of type A that might contain elements of S in some coordinates, for some $c \in \mathbb{R}$ large enough, the type of the point $p^* = p + c\sum_{i \in S} e_i$ will not contain any element of S. As p and p^* are projected onto the same point by the geometric contraction, we have that the types in the geometric contraction in directions S are indeed equal to $M_{/S}$. ☐

Theorem 34. *All tropical hyperplane arrangements of n hyperplanes in general position in $\mathbb{TP}^d$ have the same f-vector. Specifically, the number of faces of dimension k is equal to $\binom{n+d-k-1}{n-1}\binom{n+d}{k}$.*

Proof. We consider the set of all geometric contractions of the arrangement. Each region is either bounded in the original arrangement or is bounded in exactly one geometric contraction, namely the contraction in all of the region's unbounded directions. This defines a bijection between the bounded regions in the geometric contractions and unbounded regions in the original arrangement. We can therefore count all k-dimensional cells by just counting all $k-|S|$-dimensional bounded cells of all contractions plus the bounded cells in the original arrangement. For d-dimensional arrangements defined by n points in general position, all contractions are in general position, so according to Corollary 31 they have the same f-vector. Using Vandermode's identity, we obtain for the number of regions:

$$\begin{aligned}\sum_{S\subset[0..d]}\binom{n+d-k-1}{n-(k-|S|)-1,\ d-k,\ k-|S|} &= \sum_{i=0}^{d}\binom{d+1}{i}\binom{n+d-k-1}{n-(k-i)-1,\ d-k,\ k-i}\\ =\sum_{i=0}^{k}\binom{d+1}{i}\binom{n+d-k-1}{d-k}\binom{n-1}{k-i} &= \binom{n+d-k-1}{d-k}\sum_{i=0}^{k}\binom{d+1}{i}\binom{(n+d)-(d+1)}{k-i}\\ =\binom{n+d-k-1}{d-k}\binom{n+d}{k} &\quad .\end{aligned}$$

□

4.2 k-Sets

For an n-point set P, a k-point subset $S \subseteq P$ is called a **k-set** if there exists an open halfspace that contains S, and its open complement contains $P \setminus S$.

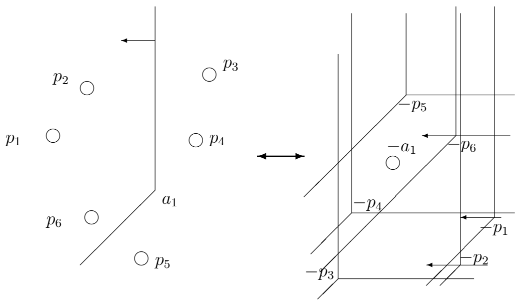

Figure 4: A k-set in $\mathbb{TP}^2$ and its dual

Upper Bound. In order to upper bound the number tropical k-sets for a point set P, we use the simple duality from Observation 6. Consider the dual hyperplane arrangement of P. A set $S \subseteq P$ of size k is a k-set if we can partition the directions $[0..d]$ into D_1 and D_2 such that a point q exists that only lies in closed sectors D_1 of any hyperplane dual to a point $s \in S$, and only lies in closed sectors D_2 of any hyperplane dual to any other point (See Figure 4). In the tropical matroid interpretation, this means that S is a k-set if there exists a type of which k components are subsets of D_1 and all $n-k$ other components are subsets of D_2. We say that such a type *realizes* S. Due to continuity, any type that realizes a k-set S has a neighboring full-dimensional region (tope) that also realizes S. That means that for each k-set, there exists one or several topes that realize it. For upper bounding the number of topes, we use the following Lemma:

Lemma 35. *Let $A = (a_1, \dots, a_n)$ be a tope in a hyperplane arrangement $\mathcal{H}_{p_1}, \dots \mathcal{H}_{p_n}$ in $\mathbb{TP}^d$. For each permutation π of $[n]$ we let $A_\pi := (a_{\pi(1)}, a_{\pi(2)}, \dots, a_{\pi(n)})$. Then for each A_π, either $A_\pi = A$ or $A_\pi \notin M$.*

Proof. If all a_i are equal, then obviously $A_\pi = A$ must hold. So we assume that not all a_i are equal, and $A_\pi = A$ does not hold. Let $p_i = (p_{i,0}, \dots, p_{i,d})$, and let x be a point with type A and y be a point with type A_π. Then for $1 \leq i \leq n$, since x lies in the open sector a_i of hyperplane $\mathcal{H}_{p_i}$, we have that $x_{a_i} + p_{i,a_i} \leq x_{a_{\pi(i)}} + p_{i,a_{\pi(i)}}$, where equality holds iff $a_i = a_{\pi(i)}$. Further, y lies in the open sector a_{π_i} of hyperplane $\mathcal{H}_{p_i}$, so $y_{a_{\pi(i)}} + p_{i,a_{\pi(i)}} \leq y_{a_i} + p_{i,a_i}$, where equality again only holds iff $a_i = a_{\pi(i)}$. Consequently, $x_{a_i} - y_{a_i} \leq x_{a_{\pi(i)}} - y_{a_{\pi(i)}}$, and equality holds iff $a_i = a_{\pi(i)}$. Since $A_\pi \neq A$, there exists at least one element i for which $a_i \neq a_{\pi(i)}$. Such an element obviously lies in a permutation cycle, so applying the last inequality on each element of this cycle would give a inequality series that adds up to $x_{a_i} - y_{a_i} < x_{a_i} - y_{a_i}$. □

To count the number of k-sets that can be separated by halfspaces consisting of just *single sectors*, we fix a direction $i \in [0..d]$ corresponding to this single sector. Then the corresponding partition of the directions is very simple: $D_1 = \{i\}, D_2 = [0..d] \setminus \{i\}$. For a fixed direction, each tope may only realize one k-set. Hence, there are at most as many k-sets for fixed direction i as there are topes for which k components are equal to i. We compute the number of topes with help of Lemma 35. Keeping fixed the k components that are equal to i, we can choose the remaining $n-k$ components from $(d+1)-1$ different values. Without permutations, this yields $\binom{n-k+d-1}{n-k}$ possibilities. If we now add up all directions, we get an upper bound of $(d+1)\binom{n-k+d-1}{n-k} \in O((n-k)^{d-1})$ on the number of k-sets which can be realized by single sectors.

For the number of k-sets that can be separated by *general halfspaces*, we proceed similarly: Let us fix $D_1 \subset [0..d]$, and sum up over all possible D_1. We note that for fixed D_1, any tope can again only realize one k-set. Therefore we can limit the number of k-sets for a fixed direction by the number of topes. Let $|D_1| = j$. Using Lemma 35, we again fix the k components which are in D_1, and count the number of possibilities for the other $n-k$ components which are in $D_2 = [d] \setminus D_1$. Using the same counting technique as above, we get at most $\binom{k+j-1}{k}\binom{n-k+d-j}{n-k}$ topes. Summing up over all possible $D_1 \subset [0..d]$, we obtain

$$\begin{aligned} \textstyle\sum_{D_1 \subset [0..d]} \binom{k+|D_1|-1}{k}\binom{n-k+d-|D_1|}{n-k} &= \textstyle\sum_{j=1}^{d} \binom{d+1}{j}\binom{k+j-1}{k}\binom{n-k+d-j}{n-k} \\ < \textstyle\sum_{j=1}^{d} 2^{d+1}\binom{k+j-1}{k}\binom{n-k+d-j}{n-k} &= 2^{d+1}\binom{n+d}{n+1} \le 2^{d+1}\frac{(n+d)^{d-1}}{(d-1)!} \in O(n^{d-1}) \end{aligned}$$

for $n > d$, which we may assume, since otherwise every k-set is realizable (see our above discussion of Radon's Lemma, Corollary 24).

Lower Bound. For the lower bound on the number of k-sets, we present two constructions, only considering halfspaces consisting of a single sector.

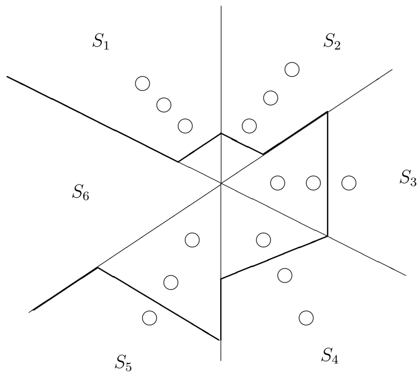

Figure 5: A sketch of construction I for 15 points in $\mathbb{TP}^5$. The bold line depicts a halfspace realizing a 5-set.

Construction I. This construction achieves $\Omega(k^{d-1})$ many k-sets for $k < \frac{n}{d}$. We choose n so that $\frac{n}{d}$ is an integer. For $1 \le i \le d$ and $1 \le r \le \frac{n}{d}$, we let $P = (p_1, \ldots, p_n)$, where $p_{rd+i} = -r\, e_i$. If we consider the hyperplane $\mathcal{H}_a$ where $a = (a_0, \ldots, a_{d-1}, -\epsilon)$ for some $0 < \epsilon < 1$ and integral $0 \le a_i \le \frac{n}{d}$. For each point p_{rd+i}, the d-th component of $a + p_{rd+i}$ is $-\epsilon$. The only other component that may be negative is the i-th, which is negative iff $a_i < r$. In this case, this is the minimal component, since $a_i + -r$ is an integer. This means that there are $\sum_{i=0}^{d-1} a_i$ points in sector d of $\mathcal{H}_a$. By varying the choice of vector a, we get a different point set in sector d of $\mathcal{H}_a$. So, there are at least as many k-sets in P as there are integral vectors $(a_0, \ldots, a_{d-1})$ such that $0 \le a_i \le \frac{n}{d}$ and $\sum_{i=0}^{d-1} a_i = k$. These are difficult to count if $k > \frac{n}{d}$, but since we assumed $k \le \frac{n}{d}$, we obtain that there are $\binom{k+d-1}{d-1} \in \Omega(k^{d-1})$ many k-sets in this setting.

Construction II. For the case $k > \frac{n}{d}$, we proceed analogous to Construction I, with the exception that we place j points along the ray $\{\lambda(-e_d) | \lambda > 0\}$. The remaining $n-j$ point are split up as in Construction I. We choose j such that

$$\begin{aligned} j + \frac{n-j}{d} \ge k \ge j + \frac{n-j}{d} - 1. \\ \Leftrightarrow (d-1)j + n \ge kd \ge (d-1)j + n - d \\ \Leftrightarrow \frac{kd-n}{d-1} \le j \le \frac{(k+1)d-n}{d-1}. \end{aligned}$$

Such an integer j obviously exists, since the gap between the lower and the upper bound is greater than 1. For $k > \frac{n}{d}$, j is positive. We look at similar hyperplanes as in Construction I. We only consider k-sets that contain all j points of sector d. Of the other sectors, each such k-set might contain $k-j$ points in total, so for each direction $i \in [0..(d-1)]$, a k-set may contain between 0 to $k-j$ points from sector i. Now since $k - j \ge \frac{n-j}{d} - 1 \ge \frac{n - \frac{(k+1)d-n}{d-1}}{d} - 1 = \frac{n-k-1}{d-1} - 1$, the number of k-sets that we count this way for $d \ge 2$ is

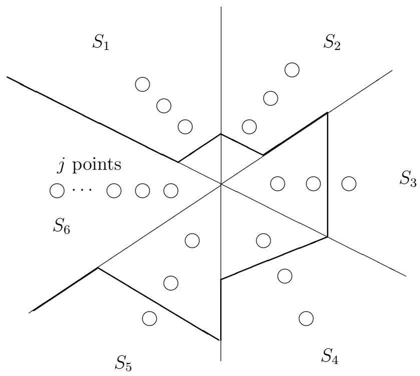

Figure 6: A sketch of construction II in $\mathbb{TP}^5$. The bold line depicts a halfspace realizing a $(5+j)$-set.

$$\binom{\frac{n-k-1}{d-1} + d - 2}{d-1} \ge \left(\frac{n-k}{(d-1)^2}\right)^{d-1} \in \Omega((n-k)^{d-1}).$$

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A Proof of the Tropical Colored Tverberg Theorem

For the proof of Theorem 5, we need the following Lemma, which is a slight variation of the Centerpoint Theorem:

Lemma 36. *For kd points in general position in $\mathbb{TP}^d$ there exists a hyperplane $\mathcal{H}$ such that each open sector of $\mathcal{H}$ contains k points.*

Proof. We introduce a new fictional point q and then apply tropical Centerpoint Theorem 2 on $P \cup \{q\}$. This implies that there exists a point c containing $k+1$ points in every closed sector of $\mathcal{H}_{-c}$. Let us look at the bipartite graph $G = (V, E)$ with $V := [0..d] \uplus (P \cup \{q\})$ and $E := \{\{i, p\} \mid p \text{ lies in closed sector } i \text{ of } \mathcal{H}_{-c}\}$. As we have shown in the proof of tropical Tverberg Theorem 3, for a centerpoint this graph is a tree, where every "left" vertex $i \in [d]$ has degree $k+1$. We now want to delete vertex q and a some edges, so that every left vertex has degree k, but every right vertex still has degree 1. As G is a tree, this is feasible as follows: If we root the tree at q and let q lie on the 0-th level of the tree, then all points lie on even levels and dimensions lie on odd levels. We delete all edges leading down from even levels. Since every node has exactly one parent, each point remains incident to an edge (except for q) and each dimension only loses one edge by the deletions, so still has degree k.

Let us see what the geometrical equivalent to deleting these edges is. Let $2l$ be the number of levels of the tree. This number always is even, as all leaves are points and are on the right side of the tree. We choose $\epsilon_1 > \epsilon_3 > \epsilon_5 > \ldots > \epsilon_{2l-1} = 0$, where ϵ_1 is smaller than the distance of any point in $P \setminus \mathcal{H}_{-c}$ to $\mathcal{H}_{-c}$. We let the apex of $\mathcal{H}$ be at

$$c - \sum_{i=1}^{d} \epsilon_{\text{level of } i \text{ in } G} e_i.$$

Because the ϵ_i are sufficiently small, no point may lie in a sector of $\mathcal{H}$ in which it does not lie for $\mathcal{H}_{-c}$. Further, each point lying on the hyperplane $\mathcal{H}_{-c}$ lies in only one sector of $\mathcal{H}$, namely in the one of its parent in the tree G. So if we draw the same bipartite graph for the apex of $\mathcal{H}$ as we did for c, we get graph described above, which has the desired properties. $\square$

Proof of the tropical colored Tverberg Theorem 5. Using Lemma 36, we find a hyperplane $\mathcal{H}$ such that every open sector of $\mathcal{H}$ contains k points. We now want to show how to choose P_i such that the apex of $\mathcal{H}$ is in the tropical convex hull of each P_i. According to Proposition 15, any P_i that contains the apex must contain at least one point from each sector of the hyperplane. Since there are only t points in a sector, each P_i contains exactly one point of each sector. The following claim implies that these P_i can be chosen as desired:

Claim 1. *Given a $t \times d$ grid of tiles that are colored with d colors such that always t tiles are colored with the same color. Then we can rearrange the tiles by permuting within the columns so that each row contains each color (is rainbow).*

Proof of claim. Proof by induction. For $t = 1$, the claim is obvious. For a fixed t, we only need to prove that the first row can contain each color. The rest follows by induction, as we may then ignore the top row when permuting. We assume for sake of contradiction that the top row can not contain tiles of each color by permuting elements within columns. Let the top row contain a maximal number of different colored tiles. We look at the directed graph $G = (V, E)$, where V is the set of colors and
$E = \{(x, y) \mid y \text{ is in the top row of a column that also contains } x\}$. A directed path from a color missing in the top row to a color that occurs more than once would yield an improvement in the number of colors on the top row and, due to our assumption, must not exist. So, starting from some missing color x, we can only reach $k < d$ vertices (including x). Then these k colors must be fully contained in $k-1$ columns, which is impossible, since there are tk blocks of these k colors, but only $t(k-1)$ blocks in these columns. $\square$

Now the connection between the claim and Theorem 5 is easy to see: Each column of blocks corresponds to a sector and a single block corresponds to a point. The rows will thus determine our desired sets P_i. $\square$